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Self-quenched dynamics

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Abstract. We introduce a model for the slow relaxation of an energy landscape caused by its local interaction with a random walker whose motion is dictated by the landscape itself. By choosing relevant measures of time and potential this *self-quenched* dynamics can be mapped on to the "True" Self-Avoiding Walk model. This correspondence reveals that the average distance of the walker at time t from its starting point is $R(t) \sim \log(t)^{\gamma}$, where $\gamma = 2/3$ for one dimension and $1/2$ for all higher dimensions. Furthermore, the evolution of the landscape is similar to that in growth models with extremal dynamics.

PACS. 05.40.Fb Random walks and Levy flights – 05.65.+b Self-organized systems – 05.50.+q Lattice theory and statistics (Ising, Potts, etc.)

1 Introduction

The motion of random walkers in a random environment is one of the basic problems in the physics of disordered systems [1–3]. It is known that the effect of the environment on the walker results in anomalous diffusion in some cases and logarithmically slow diffusion in others [1,2]. Apart from their intrinsic interest, these simple models also find applications in several physical processes, such as the diffusion of electrons in a disordered medium [4] or glassy activated dynamics [3]. In particular, the Sinai model [2] has been extensively studied from this point of view. It is known that for this model, the walker becomes logarithmically slow, moving as $R(t) \sim \log^2(t)$ where $R(t)$ is the average distance at time t of a walker from the launching point. Apart from this, two-time aging dynamics studied in this model also provide close analogies to glassy phenomenology [3].

In another framework, a "trapping" model ([5] and references therein) was introduced to provide a simple example of the glass transition. A single walker explores a random landscape of energy traps with *e.g.* an exponential distribution, from which it can escape through activated hops. At low temperatures, it was shown that the system cannot reach a steady state due to the unfavorable competition between the depth of the visited traps and the time needed to escape from them. The corresponding slow dynamics, and disappearance of an equilibrium state was interpreted as a glass transition. In this mean-field model, introducing an interaction between the walker and the random energy landscape was shown to not change the results [5]. However, the fact that the trapping time distribution is a power-law provides a natural (statistical) history dependence. (The dynamics is controlled by the deepest energy well visited so far.) In the model we introduce here, the walker modifies its environment; aging and slow dynamics result purely because of this interaction.

Active walker models where the walker and the environment mutually affect each other have been studied earlier in different contexts. One such model — the Eulerian Walkers Model (EWM) — has been studied [6] within the framework of self-organized criticality (SOC) [7]. The "landscape" here is defined by an arrow at each site. The walker follows the direction of the arrow at its site after which the direction of the arrow is changed according to some fixed rules. Besides many correspondences between the EWM and the Abelian Sandpile Model of SOC [8], it was also shown that the motion of the walker was subdiffusive in two dimensions, *i.e.* $R(t) \sim t^{1/3}$. In one dimension, $R(t) \sim t^{1/2}$ due to a very simple organization of the landscape under the rules. For $d > 2$ it was argued that the walker is diffusive. Models of mutually interacting walkers and landscape have also been studied extensively in the context of pattern formation and biophysical applications, where the emphasis is on the patterning of the medium under the influence of multiple walkers [9,10].

The Self-Avoiding Walk (SAW) [11], where the walker is obliged to avoid its former path, and its modifications, can also be regarded as active walks in the sense that the path of the walker in these models is influenced by the trace it has made. For later purposes it is useful to mention here the so called "True" Self-Avoiding Walk (TSAW)

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where the walker's probability to go to an already visited site is a strongly decaying function of the number of visits to that particular site [12].

The aim of this paper is to study a simple model of an active walker which exhibits *logarithmically* slow dynamics entirely due to the local interaction of the walker with a self-created random environment – a kind of self-organized trapping or self-burying effect. Besides the above mentioned connections with the physics of glasses, the model in one dimension may also be regarded as a very simplified version of a recently introduced model for the slow dynamics of sheared granular media, where the wandering of shear bands and the related restructuring of the material was shown to lead to extremely slow relaxation processes and inhomogeneous aging [13].

2 Definition of the SQW model

Our model is defined as follows. A walker can move on a hyper-cubic lattice in a d dimensional space with linear size L and $N = L^d$ number of sites. Periodic boundary conditions are imposed. A variable s_i chosen from a uniform random distribution $[0, 1]$ is initially assigned to every site i on the lattice. We call the site with the walker the *active* site.

The only permitted elementary moves for the walker are to the nearest neighbours. At every time step a new random number $s_i(t)$ (uniformly sampled between 0 and 1) is assigned to the active site i. If this random number is larger than the value of the s variable of *all* the nearest neighbours, then the same site remains active. Otherwise, the activity moves to the neighbouring site with the largest value of s and the same procedure is repeated. We have chosen here a uniform distribution for s, however, this specific form can be shown to play no role in the time evolution of the active site. In the following we will refer to this model as a Self-Quenched Walk (SQW).

As time goes on, the average value of s decreases and as a result, the probability that the activity moves to one of the neighbours decreases. The definition of the model is thus quite inconvenient in terms of numerical simulation since intervals when nothing happens grow longer and longer with time. However, it is easy to circumvent this difficulty. Let σ be s_j, the largest s value amongst the neighboring sites of the active one, i. For the activity to move to j, the s_i value has to be smaller than σ which is an event with probability σ. Thus the waiting time before a move is a Poisson process with a characteristic time $\tau = 1/\sigma$. Once $s_i < \sigma$, the evolution of the activity is deterministic. Moreover, the distribution of s_i when the activity moves to j is uniform between 0 and σ. Therefore, we can directly reproduce the evolution of the model in terms of the number of moves n rather than in time t. It turns out that the variable n is also more convenient for the analytical treatment.

Similarly, since the s values quickly evolve towards 0, as t increases, it is more convenient to use an equivalent parameterization introducing $r \equiv -\log(s)$. The uniform distribution of s between 0 and σ implies that r is distributed with a density $\mathrm{e}^{\rho - r}$ for $r > \rho = -\log(\sigma)$. Alternatively, we note that $r - \rho$ is a random variable exponentially distributed from 0 to ∞. This reformulation allows simulations to be carried out over practically unlimited times without loosing any accuracy. Moreover, as we will see below, r is the appropriate scale for providing an accurate description of the long time regime.

Let us thus consider the motion of the activity (the walker) as a function of the number of moves n, in a potential $V(i, n)$ (where the value of the potential $V(i, n) = r_i(n)$). The walker moves to the neighbouring site with the *smaller* value of r [since $r \equiv -\log(s)$] after having changed the value of V on the site it was on. The above statement can be made more quantitative in the following two coupled equations of evolution for the walker and the r-landscape:

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}n} = -\nabla V(\mathbf{X}(n), n) + \eta(n) \tag{1}$$

$$\frac{\partial V(\mathbf{x}, n)}{\partial n} = \lambda \delta^d(\mathbf{x} - \mathbf{X}(n)) \tag{2}$$

where $\langle \eta(n) \rangle = 0$ and $\langle \eta(n)\eta(n') \rangle = C\delta(n - n')$. Here $\mathbf{X}(n)$ is the position of the walker at 'time' n.

Equation (1) quantifies the rules of the SQW in any dimension. When the walker is on a slope, it moves down towards the valley. When it is in a flat region, it moves to any of the nearest neighbours with equal probability (this is the reason for the uncorrelated noise term). Equation (2) accounts for the increase of the r-landscape at the position of the walker. In this simplified continuum description, we have neglected the randomness in the distribution of the local increments in r, and only retained the average value λ. However, as we shall see below, all the essential features of the problem are contained in the equations.

3 Correspondence with the TSAW model

It turns out that the Langevin equations (Eqs. (1, 2)) of the SQW in terms of n and r are the same as that of the TSAW as a function of time and position [12]. The definition of the TSAW model is the following: The walk takes place on a d-dimensional hypercubic lattice. At any step the traveller may move to any of the $2d$ nearest neighbours of the lattice site he is at. The probability of stepping to site i depends on the number of times n_i this site has already been visited and is given by

$$p_j = \exp(-g n_j) \left[\sum_{j=1}^{2d} \exp(-g n_j) \right]^{-1}, \tag{3}$$

where the sum runs over all $2d$ nearest neighbours of the current position of the walker, and g is a positive parameter which measures the intensity with which the walk avoids itself. Note that the sum over i of p_i is equal to 1, meaning that the traveller never stays at the same site.

This is similar to the SQW when time is incremented in units of n.

In the SQW the value of s decreases exponentially on average since each time the walker visits the site we multiply the s value by a random number taken from a uniform distribution between 0 and 1. The value of $r = -\log(s)$ thus increases linearly with the number of times the traveller has visited this site.

Further, in the SQW, whenever the walker can move, it goes deterministically to the neighbouring site with the higher value of the potential. This is realized by the $g \to \infty$ limit of the TSAW model.

The above mapping thus ensures that in the continuum limit the two models are governed by the same Langevin equations. The role of the preassigned probability (Eq. (3)) in the TSAW is played by the self-organized evolution of the walker and the potential.

The TSAW has been studied exhaustively by means of numerical simulations [14,15], Flory theory [16], scaling analysis [17] and later by exact calculations [18].

The critical dimension of the TSAW problem is $d_c = 2$ above which the mean field solution applies and the traveller's asymptotic behavior is not influenced by the interaction with its former path and performs basically a Brownian motion. Below two dimensions the trace of the walker is a *fractional* Brownian motion. The root mean square distance from the origin increases as

$$\langle |\mathbf{X}(n) - \mathbf{X}(n')|^2 \rangle^{1/2} \propto |n - n'|^{\nu} \tag{4}$$

with

$$\nu = \begin{cases} 2/(d+2) & \text{for} \quad d \leq 2 \\ 1/2 & \text{for} \quad d > 2. \end{cases} \tag{5}$$

Thus in one dimension the walker is super-diffusive with a Hurst exponent of $\nu_{d=1} = 2/3$ which is due to the repulsive interaction with its former path. This asymptotic behavior is numerically verified for the SQW in Figure 1.

We can define other related exponents through the scaling relations: $X \to bX$, $n \to b^z n$ and $V \to b^{\chi} V$, where χ is the so called *roughness exponent* of the r landscape and z is the dynamic exponent. Equations (1) and (2) then predict the following values for these exponents:

$$\begin{array}{llll} \chi = 1/2 & z = 3/2 & \text{in} & d = 1 \\ \chi = 0 & z = d & \text{for} & d \geq 2. \end{array} \tag{6}$$

We have confirmed the value of χ and z by measuring the width of the r landscape in our model.

4 Roughness of the potential

The width of a self-affine interface is defined as the root mean square fluctuation of the interface from its mean value. In a number of growth models, this width obeys

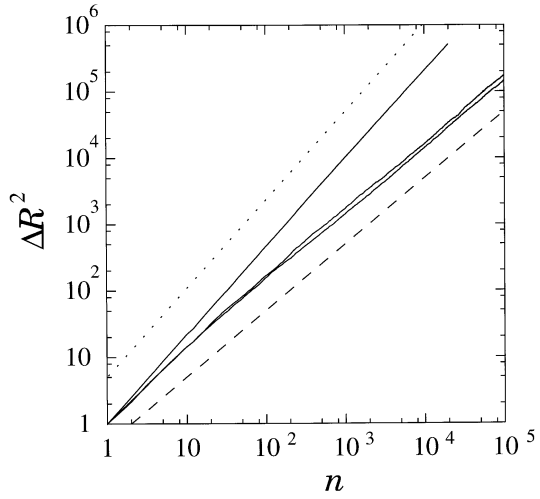

Fig. 1. The mean square distance from the origin covered by the walker in the steady state, as a function of the number of moves n. The dotted line shows power law with exponent 4/3 (upper) and the dashed with an exponent 1 (lower). The curves are for dimensions 1, 2 and 3 from top to bottom respectively.

the Family-Vicsek scaling [19] with a dynamic exponent z such that the overall roughness follows

$$w(n) \propto L^{\chi} \varphi\left(\frac{n}{L^z}\right). \tag{7}$$

Figure 2 shows our numerical determination of the width of the r landscape for five different system sizes in one and three dimensions. The collapse with the above mentioned value of the exponents indicates that the r landscape is self-affine in one dimension. However, the growth exponent β describing the roughening of the landscape fluctuations for early times ($n < n^* \sim L^z$) as $w \sim n^{\beta}$ does not obey the Family-Vicsek scaling $\beta = \chi/z$. It is instead given by

$$\beta = (\chi + 1/2)/(1 + \chi), \tag{8}$$

a formula typical for growth models with extremal dynamics [20]. Thus our model, though it does not contain a global extremum criterion, belongs to the class of extremal growth models.

The reason why the interface growth (or the r landscape evolution) is similar to extremal dynamics is the following. The activity is a 'random' walk trapped by the maxima of the r-landscape. Before the activity can escape, the landscape has to be filled. In order to escape from a region of extent $\tilde{l}$, the number of moves to be made is of the order of the size of the valley, $\tilde{l}^d$, times its typical depth $\tilde{l}^{\chi}$. The growth is pinned everywhere except in the immediate vicinity of the walker. The walker itself is, however, in a hierarchically ordered valley structure with maxima of increasing heights. Thus the interface progresses jerkily just as in other extremal growth models. The relation $z = d + \chi$ predicted by equation (2) is also known to occur in various extremal growth models [21,22].

The correspondence with extremal dynamics does not hold above $d_c = 2$. This is because above two dimensions,

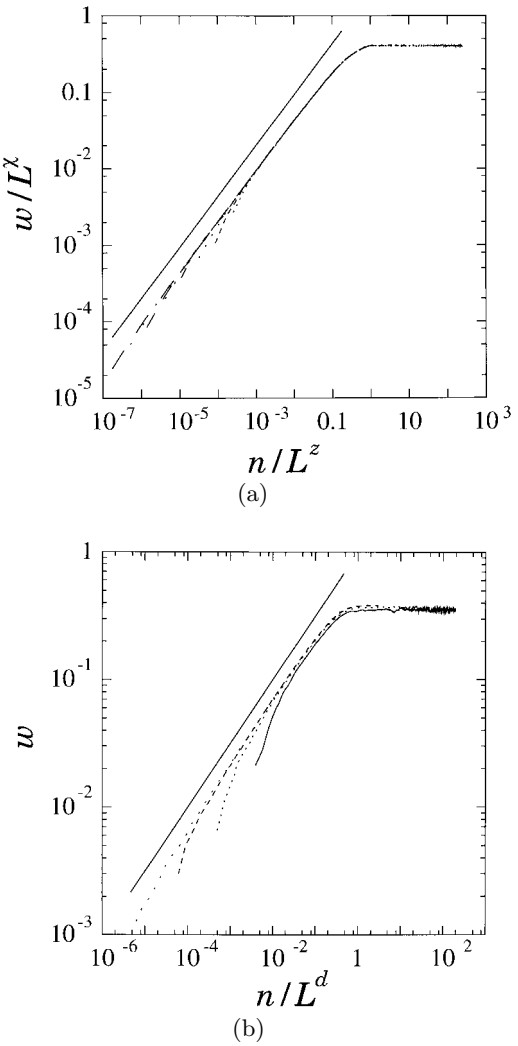

Fig. 2. a) Average of squared roughness w in one dimension scaled by L^{χ} as a function of the number of moves n scaled by the dynamic exponent L^z. The different curves refer to system sizes 256*1, 2, 4, 8, 16; each averaged over 1000 realizations. The slope of the solid line is $\beta = 2/3$. b) In 3 dimensions the steady state value of the width does not depend on the system size. System sizes 8, 16, 32, 60 were used each averaged over 1000 realizations. The slope of the solid line is $\beta = 1/2$.

the walker is no longer trapped by surrounding maxima. It can also find its way around them instead of over them. As a result the interface is no longer rough and $\chi = 0$. However, though the width saturates to a system size-independent value, the exponent β is non-zero for the following trivial reason. In the random initial state, in n steps, a number n sites grow by, say, a uniform amount h. Therefore the width w scales trivially as $w \sim hn^{1/2}L^{-1/2}$. The exponent β is hence equal to $1/2$ for all $d \geq 2$.

5 Real time behaviour

We now turn to the behaviour of the walker in real time. In order to understand this, we first note that the mean

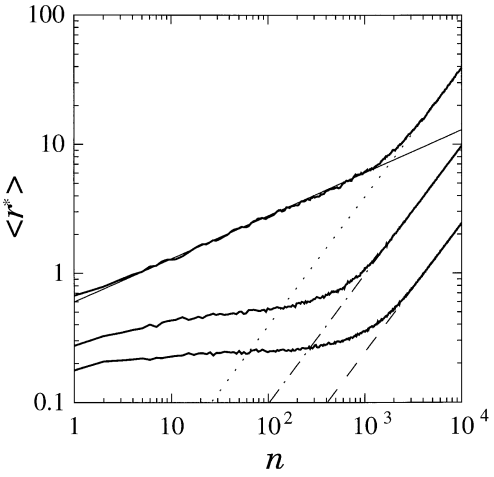

Fig. 3. Average of r^* as a function of the number of moves n, for a system of size $L = 256$, $L = 32$ and $L = 16$ in 1, 2 and 3 dimensions respectively from top to bottom averaged over 1000 samples. The solid line shows the a power-law of exponent 1/3, whereas the dashed lines are the asymptotic exact increase $r^* = n/N$.

value of r in the steady state increases linearly with the number of moves. Hence the mean increase of r per site in the steady state is $1/N$, where $N = L^d$ is the number of sites in the system.

We now compute $\langle r^*(n)\rangle$ (the r value of the active site as a function of n) and observe two regimes that we describe by the scaling assumption:

$$\langle r^*(n)\rangle \propto n^{\alpha}\psi\left(\frac{n}{L^z}\right) \tag{9}$$

with $\psi(a) \propto a^0$ for $a \ll 1$ and $\psi(a) \propto a^{1-\alpha}$ for $a \gg 1$. Therefore, for long times $\langle r^*(n)\rangle \propto nL^{-z(1-\alpha)}$. However, in the long time regime, r^* has to increase at the same rate as the mean velocity of the front $\langle r\rangle$, and hence, for $n \gg L^z$, $\langle r^*(n)\rangle = n/N = nL^{-d}$. This imposes

$$z(1-\alpha) = d \quad \text{or} \quad \alpha = 1 - \frac{d}{z} = \frac{\chi}{d+\chi} \tag{10}$$

i.e. $\alpha = 1/3$ for $d = 1$ and 0 for $d \geq 2$ consistently with our numerics as shown in Figure 3.

Since, in the late stage regime, r^* increases as n/N, s^* decreases as $\mathrm{e}^{-n/N}$. Therefore the expectation value of the real time lapse between two consecutive moves $t(n) - t(n-1)$ is $1/s^*$ or

$$t(n) - t(n-1) \approx \mathrm{e}^{n/N} \tag{11}$$

$$t(n) = \frac{\mathrm{e}^{(n+1)/N} - 1}{\mathrm{e}^{1/N} - 1} \sim N\mathrm{e}^{n/N}. \tag{12}$$

This law refers in fact only to the mean value of t. However, the distribution of each increment being exponentially distributed (a Poisson process), the central-limit

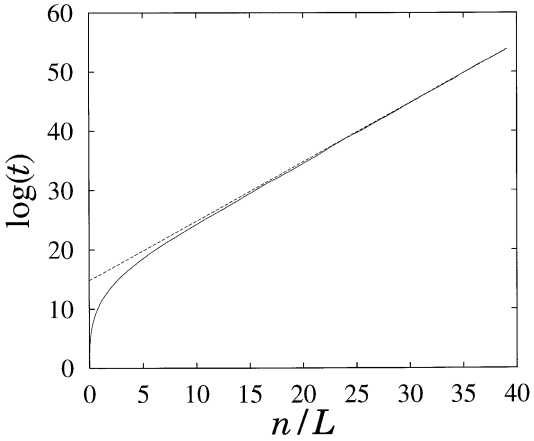

Fig. 4. Average of the logarithm of the time as a function of the number of moves in $d = 1$, scaled by the system size n/L. The system size is $L = 256$ and the average is performed over 1000 samples. The dotted line is a line of slope 1, as theoretically expected.

theorem applies, and the relative standard deviation of t with respect to its mean value vanishes. Figure 4 shows the numerically determined time as a function of the number of moves n.

The average value of s as a function of time is thus

$$\langle s \rangle \propto \exp(-n/N) \propto \exp(-\log(t/N)) \propto N/t \qquad (13)$$

for long times.

Therefore in real time the walker is logarithmically slow with $R(t) \sim \log(t)^{2/3}$ in one dimension and $R(t) \sim \log(t)^{1/2}$ in higher dimensions. The logarithmic dependence of the RMS distance of the walker is just the consequence of equation (13) as a result of which, the probability of making a jump to a neighbouring site decreases as $1/t$. It is thus valid in any dimension. The value of the exponent of the log in one dimension, is however a non-trivial consequence of the coupling between the walker and the medium which induces long-range memory effects.

6 Conclusion

In summary, we have introduced and studied a simple model of a walker interacting with its environment. By choosing the correct measures for describing the time and the potential, we could map the SQW problem to the TSAW model and thus use the known exact results in the latter case, to describe the motion of the SQW walker. In addition, we have also studied the emerging landscape. Though the rules for the SQW are entirely local, a relationship with the so called extremal models could be established. The critical dimension is $d_c = 2$ below which the potential landscape gets self-affine and the walker superdiffusive in terms of moves. In real time the walker is logarithmically slow in any dimension.

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